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Some Analytical Solutions for Groundwater Flow and Transport Equation

EFFAT A. SAIED and M. E. KHALIFA
Department of Mathematics, Faculty of Science, Benha University, Benha, Egypt

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Abstract. This paper presents the use of symmetry reduction method resulting in new exact solutions for the groundwater flow and transport equation. It is assumed that the radionuclides are transported by advection-diffusion in a single fracture and diffusion in the surrounding rock-matrix. The application of one-parameter group reduces the number of independent variables, and consequently the governing PDE of $(1+2)$-dimension reduces to set of ODEs which are solved analytically. This enables us to present some new exact time-dependent solutions of the advection-diffusion equation.

Key words: advection-diffusion systems, group properties of partial differentia1 equations (PDEs), similarity reduction.

1. Introduction

The phenomenon of transport of flow through a porous medium is relevant in numerous areas. ‘Groundwater,’ for example, a term used to describe the flow of water through the Earth’s crust, has long been an area of intense study. The movement of pollutants through groundwater is of increasing importance in considering alternatives for disposing of nuclear waste (Dagan, 1987). In the event of a repository failure, leaching of nuclides by water as well as nuclide migration in the bedrock is of major significance from a consequential point of view. There is some evidence, that the buffer may be susceptible to cracking due to many causes. Consequently, the preventive properties of the buffer with respect to such cracks are important and several related studies of mass transport through cracks are available in the literature (Rasmuson and Neretnicks, 1981; Tang *et al.*, 1981; Sudiscky and Frind, 1982; Chen, 1986).

The migration of radionuclide has been extensively modeled. The model equations accounting for the effects of matrix diffusion on transport in fissured media are given by Barker (1982, 1985), where the analytic solutions have been obtained using Laplace transformation and for the redial dispersion in double-porosity aquifer fracture is given by Mochnch (1995). Other research in this area by Maloszewski *et al.* (1990, 1993) and Cvetkovic *et al.* (1999) presents mathematical modeling of tracer behavior in fractured rocks.

The starting point for analytically describing the complicated porous media flow of a species is the advection-diffusion equation (Bear, 1979);

$$\frac{\partial C}{\partial t} = -\nabla . J, \tag{1}$$

where flux, J, is

$$J = -D\nabla C + vC, \tag{2}$$

with $C(r, t)$ denoting particle concentration, D being the diffusion coefficient and v representing the flow velocity. This equation gives the transport due to advective motion. When D and v are constants $C(r, t)$ satisfies the PDE

$$C_t = D\Delta C - v\nabla C. \tag{3}$$

In the steady state, Equation (3) was studied by Phillip *et al.* (1989). They were considering the flow of groundwater around a cylindrical obstacle. They obtained an exact solution for $C(r)$ as an infinite series, by using separation of variables. In an effort to better understand the problem of radionuclides transport through a cracked buffer of a repository, in order to determine the effect of cracks on the fuel dissolution rate, Garisto *et al.* (1990) have presented a detailed description of some related models of Equation (3). The equations for these transport models have been solved by numerical methods. Barten (1996) discussed the common advection-diffusion equation utilized as a transport model, and the linear response concept combining advection with limited rock matrix diffusion in a fracture network. Progress in research with these and similar devices has been impeded by the lack of an exact analytic solution of the corresponding advection-diffusion processes.

The analytic solutions of the governing equations, if available, present a quantitative and qualitative understanding of the phenomenon, and allow examining the sensitivity of the models suggested to several important physical parameters contained in the models. The similarity transformation method (Bluman and Cole, 1974; Oliver, 1986; Hill, 1992) can help in finding exact explicit solutions of Equation (3) and its special forms, Equation (4) when the advection term is omitted, and Equation (9) when the advection term is considered.

The exact similarity solutions are important because they readily identify interesting and unusual physical phenomena which may be quite novel and unexpected and which may be hard to identify or may not be quite so transparent from a numerical solution of the governing partial differential equation. A procedure to obtain new exact solutions by similarity method has also been successfully applied to some different physical partial differential equations under potential symmetries (Khalifa *et al.*, 1994; Saied, 1999). The fundamental basis of the technique is that, when a PDE is invariant under a Lie group transformations, a reduction transformations exist. With the help of these transformations, first, the $(1+2)$-dimension PDE is reduced to system of PDEs of two new variables, say s, z and dependent function $F(s, z)$ and require that $F(s, z)$ satisfies the original PDE. Secondly, we look for reductions of these PDEs to ODEs of $G(r)$, where $r = r(s, z)$. This imposes conditions upon the infinitesmal functions of s, z, r and F, G and their derivatives in the form of an overdetermined system of equations, whose solutions yield the desired reductions to ODEs, which may be solvable explicitly. Solutions $G(r)$ lead to similarity solutions $u(x, y, t)$ of Equation (4) and $w(x, y, t)$ of Equation (9).

In the course of doing this, we use group theoretic approach to analyze the governing free boundary PDE (3), see Appendix, where a great variety of solutions, which may be suitable for some different physical real processes depending on whether or not they satisfy the initial and boundary conditions corresponding to each case. The paper is organized as follows; in Section (2) we express the basic equations and the probable release function of radionuclides initial and boundary conditions. Then, a two-parameter family of exact closed-form solutions is presented and the physical realization of these analytical solutions is discussed. In the Appendix, we describe briefly the symmetry group analysis of the PDE, which enables us to obtain these great varieties of solutions.

2. Problem Formulation and Closed Form Solutions

The physical geometry of the system considered in this paper consists of a single fracture (planar and finite) of width E of a repository containing canisters filled with radioactive waste. Once the repository is sealed, nuclides can only escape to the biosphere in the event of canister failure. Following this event, nuclides leach out of the canisters and migrate through the engineered barriers. Once outside the repository, the nuclides are transported by the flowing groundwater to the biosphere.

In the present study, the cracked buffer system is modeled with the following assumptions, the dissolved radionuclides diffuses in a radial direction, away from the whole length of the container, that is, end effects are neglected. The crack is narrow compared to the length and/or thickness of the buffer zone, therefore, the concentration in the crack should be uniform along the y-direction. All the engineered barriers buffers are treated as a single homogenous medium, having the same physical properties, and the migration of nuclides through the barriers is diffusion controlled. In this case, following Garisto *et al.* [14] and from Equation (3), the concentration of the pore-water in the buffer $C(r, t) \equiv u(x, y, t)$, can be given by the diffusion equation

$$u_t = M(u_{xx} + u_{yy}), \tag{4}$$

where $M = D/R_b T_b^2$, D is the pore-water diffusion coefficient or hydrodynamic dispersion, being a tensor that depends on velocity of flow in addition to dispersivity values, R_b and T_b are retardation factor and tortuosity in the buffer.

The above PDE is subject to the initial condition

$$u(x, y, 0) = 0. \tag{5}$$

The rate of transport of nuclides in geomedia by advection as well as by diffusion is expected to be significantly higher than the rate of transport by diffusion in the backfill. Therefore, the boundary conditions may be simplified to

$$u(x, y, t) = 0, \quad \text{for} \quad x^2 + y^2 \to \infty, \tag{6}$$

$$u_y = 0, \quad \text{at} \quad y = E. \tag{7}$$

At the rock/buffer interface, $x = L$ we may require flux boundary condition to be one of the following forms:

$$\frac{-D}{T_b} u_x = 0, \tag{8a}$$

$$\frac{-D}{T_b} u_x = q\gamma(t)(u(L, t) - Q), \tag{8b}$$

$$\frac{-D}{T_b} u_x = q(u(L, t) - Q), \tag{8c}$$

where q is the mass-transfer coefficient and Q is the background concentration of radionuclides in the groundwater and L is the thickness of the buffer and $\gamma(t)$ is a function to be determined.

On the other side, nuclides migrating out of the repository through the cracked buffer are transported through groundwater by diffusion as well as by advection. If v is the pore-water velocity in the crack, the migration in this case is enhanced by the flowing groundwater. The mass transport in the cracked buffer $C(r, t) \equiv w(x, y, t)$ is given by the advection-diffusion equation

$$w_t = m(w_{xx} + w_{yy}) - nw_x, \tag{9}$$

where $w(x, y, t)$ is the pore-water concentration and $m = D/R_cT_c^2, n = v/R_cT_c$, R_c and T_c are retardation factors and tortuosity in the crack. The PDE (9) is subject to the initial condition

$$w(x, y, 0) = 0. \tag{10}$$

As previously mentioned, we neglect end effects and we may use zero flux boundary condition,

$$\frac{\partial w}{\partial y} = 0 \quad \text{at} \quad y = E. \tag{11}$$

At the rock/buffer interface, $x = L$, we may require the flux boundary condition to be one of the forms:

$$\frac{-D}{T_c} w_x + vw = \delta(v)(w(L, t) - Q), \tag{12a}$$

$$\frac{-D}{T_c} w_x + vw = q(w(L, t) - Q), \tag{12b}$$

where D, T_c, v, q, and Q are defined as before, and $\delta(\nu)$ is a function of ν to be determined.

The mass-transport equations in the buffer (4) and in cracks (9) were solved analytically using the symmetry reduction method (see Appendix). We have a great variety of solutions; the formal solutions expressible in terms of elliptic functions and traveling wave solutions, however, they fail to satisfy the initial and boundary conditions of the release function of radionuclides, yet they may be suitable for some other different physical processes, so we should not omit them.

In the remaining part of this section, we will consider two families of exact solutions subject to the initial and boundary conditions which specifically address the problem of radionuclides transport through a buffer (5–8) and cracked buffer (10–12). We will obtain concentration profiles in the transformed domain in order to determine the effect of cracks on the radionuclides concentration in groundwater in the safety assessment of used fuel disposal.

2.1. THE FIRST TYPE SOLUTION

Let us consider time-dependent solution of Equation (9), representing the explicit form of the pore-water concentration due to the cracked buffer, we have

$$w(x, y, t) = \frac{A}{t} \exp\left(-\frac{(x-L)^2 + (y-E)^2}{4mt} - \frac{n^2 t}{4m} + \frac{n(x-L)}{2m}\right) - Q, \qquad (13)$$

where A is arbitrary constant. For more details of obtaining this formal solution, see Appendix; (A9).

It should be mentioned that, at $t = 0$, the background concentration of radionuclide in the ground water, Q is zero, and the exponential function in (13) tends to zero much faster than any power function, so the initial condition (10) will be satisfied by solution (13).

Differentiating the formal solution (13) with respect to the variable y, we get

$$w_y(x, y, t) = \frac{A(y-E)}{-2mt^2} \exp\left(-\frac{(x-L)^2 + (y-E)^2}{4mt} - \frac{n^2 t}{4m} + \frac{n(x-L)}{2m}\right) \qquad (14)$$

this guarantees that $W_y(x, E, t) = 0$ and the boundary condition (11) will be satisfied. Similarly, differentiating with respect to x and substituting for $x = L$, we get

$$w_x(L, y, t) = \frac{n}{2m} \frac{A}{t} \exp\left(-\frac{(y-E)^2}{4mt} - \frac{n^2 t}{4m}\right), \qquad (15)$$

where $n/m = \nu T_c / D$. Introducing the expression

$$w(L, y, t) = \frac{A}{t} \exp\left(-\frac{(y-E)^2}{4mt} - \frac{n^2 t}{4m}\right) - Q \qquad (16)$$

into (15), we have

$$\frac{-D}{T_c} w_x(L) + \nu w(L) = \frac{\nu}{2}[w(L,t) - Q]. \tag{17}$$

Thus, the flux boundary condition at the rock/buffer interface (12a) will be satisfied by the formal solution (13), and it presents some relation between the mass-transfer coefficient q and the pore-water velocity ν.

Since the pore-water velocity in buffer is zero, that is $v = n = 0$, we have the following exact solution of Equation (4):

$$u(x,y,t) = \frac{A}{t} \exp\left(-\frac{(y+\beta)^2 + (x+\alpha)^2}{4Mt}\right) + Q, \tag{18}$$

where α and β are matching parameters, they can be chosen to satisfy some required boundary conditions. For the same reasons as before the initial condition (5) at $t = 0$ will be satisfied. It is known that at infinity, that is, for $x^2 + y^2 \to \infty$, the behavior of an exponential function dominates the behaviour of any power, so that, $u(x,y,t) = 0$ as $x^2 + y^2 \to \infty$, and the boundary condition (6) is satisfied. Now, by considering the exponential derivative of the formal solution (18) with respect to variable y, we get: $u_y(x,E,t) = 0$, if we choose $\beta = -E$ in the formal solution (18), the boundary condition (7) will be satisfied if $\beta = -E$. Similarly,

$$u_x(x,y,t) = \frac{(x+\alpha)}{-2Mt} \frac{A}{t} \exp\left(-\frac{(y+\beta)^2 + (x+\alpha)^2}{4Mt}\right) \tag{19}$$

if we choose $\alpha = -L$, then $u_x(L,y,t) = 0$ satisfies the boundary conditions (8a), but if we choose $\alpha = (2MT_b q/D) - L$, then

$$u_x(L,y,t) = \frac{T_b q}{-Dt} \frac{A}{t} \exp\left(-\frac{(y+\beta)^2 + (2MT_b q/D)^2}{4Mt}\right). \tag{20}$$

Introducing the expression

$$u(L,t) - Q = \frac{A}{t} \exp\left(-\frac{(y+\beta)^2 + (2MT_b q/D)^2}{4Mt}\right)$$

into (20), one gets

$$\frac{-D}{T_b} u_x(L) = \frac{q}{t}(u(L,t) - Q. \tag{21}$$

Satisfying the flux boundary condition (8b) at the rock/buffer interface $x = L$, shows that the mass transfer coefficient q through the buffer was deeply affected by time. The added parameters α and β do not change the fact that (18) is exact solution of Equation (4) and we shall determine the parameters α and β by matching this expression with the expression required near the edges depending on the physical problem one investigating.

2.2. THE SECOND TYPE SOLUTION

The concentration of dissolved radionuclide released from used fuel through the cracked buffer can be obtained from the formal solution of Equation (9); as

$$\begin{aligned} & w(x, y, t) \\ & \quad = \frac{A}{\sqrt{t}} \exp\left\{-\frac{(y-E)^2}{4mt} + (\alpha^2 m - \alpha n)t + \alpha(x-L)\right\} + \beta Q, \end{aligned} \tag{22}$$

where α and β are matching parameters. We shall determine them by matching this formal solution with the required boundary condition. For more detail of obtaining this formal solution see Appendix (A14). The initial conditions at $t = 0$ will be satisfied by (22) for the same reasons as before. By considering the exponential derivative of solution (22) with respect to y, we get $w_y(x, E, t) = 0$ which verifies that the boundary condition (11) is satisfied.

Let us now see how this formal solution (22) can satisfy the flux boundary condition at the rock/buffer interface $x = L$. Now since

$$w_x(L, y, t) = \alpha \frac{A}{\sqrt{t}} \exp\left\{-\frac{(y-E)^2}{4mt} + (\alpha^2 m - \alpha n)t\right\} \tag{23}$$

and

$$w(L) = \frac{A}{\sqrt{t}} \exp\left\{-\frac{(y-E)^2}{4mt} + (\alpha^2 m - \alpha n)t\right\} + \beta Q \tag{24}$$

then the boundary condition (12) reads

$$\begin{aligned} & \frac{-D}{T_c} w_x(L) + \nu w(L) \\ & \quad = \left(\frac{-\alpha D}{T_c} + \nu\right) \frac{A}{\sqrt{t}} \exp\left\{-\frac{(y-E)^2}{4mt} + (\alpha^2 m - \alpha n)t\right\} + \beta \nu Q \\ & \quad = \left(\frac{-\alpha D}{T_c} + \nu\right) [w(L) - \beta Q] + \beta \nu Q \\ & \quad = \left(\frac{-\alpha D}{T_c} + \nu\right) w(L) + \frac{\alpha \beta D}{T_c} Q. \end{aligned}$$

If we choose the matching parameters α and β in Equation (22) as follows:

$$\alpha = \frac{(\nu - q)T_c}{D} \quad \text{and} \quad \beta = \frac{q}{q - \nu},$$

one gets

$$\frac{-D}{T_c} w_x(L) + \nu w(L) = q(w(L) - Q),$$

and the flux boundary conditions (12b) at the rock/buffer interface is satisfied by the solution (22).

The concentration through the buffer can be derived as a special case of Equation (22), where $n = \nu = 0$. The exact solution of Equation (4) is represented as follows:

$$u(x, y, t) = \frac{A}{\sqrt{t}} \exp\left\{-\frac{(y-E)^2}{4Mt} + \alpha^2 Mt + \alpha(x-L)\right\} + Q, \tag{25}$$

where $\alpha = -(qT_b/D)$. For the same reasons as before the initial condition (5) and boundary condition (6) will be satisfied by solution (25).

Differentiation of exponential derivative of the function $u(x, y, t)$ with respect to y at $y = E$, leads to, $u_y(x, E, T) = 0$ satisfying the boundary condition (7).

Differentiating the formal solution (25) with respect to x gives,

$$u_x(L, y, t) = \frac{-qT_b}{D} \frac{A}{\sqrt{t}} \exp\left\{-\frac{(y-E)^2}{4Mt} + \alpha^2 Mt\right\}, \tag{26}$$

and substituting by

$$u(L) - Q = \frac{A}{\sqrt{t}} \exp\left\{-\frac{(y-E)^2}{4Mt} + \alpha^2 Mt\right\},$$

into (26), one gets

$$\frac{-D}{T_b} u_x(L) = q(u(L) - Q),$$

which satisfies the flux boundary condition at the rock/buffer interface at $x = L$.

The physical sense of these exact solutions is in their describing the concentration of dissolved radionuclide released from used fuel into the buffer and crack as a function of space and time. In addition, this explicit dependence on the parameter values representing probable release function of radionuclide conditions is useful in extracting information of interest in the safety assessment of used fuel disposal. In this case, explicit calculations are required with realistic values for the parameters appearing in the transport equations. The calculations presented here are only intended to model hypothetical scenarios.

3. Conclusion

The focus of this paper has been on the development of analytical solutions of the transport equation for radionuclides in geological formations with simplified forms of the rate processes. In this processes, the transport is described by advection-diffusion in a fracture coupled to diffusion in the surrounding rock pores.

The technique of similarity solution, used here, is one of the most natural and universal tools for investigating and classifying fundamental phenomena in the exact sciences. It may be considered one of the simplest and most aesthetically justified sources of scientific prediction with claim to reliability.

We have found several new similarity reductions and explicit solutions of these reduced equations. With our calculations we have demonstrated that the symmetry reduction method can lead to an ansatz to separate independent variables. This method is very useful if we remember that there is more than one possibility for separation.

In principle, the generality of the analytical technique, with incorporation of possible special and temporal variations of the associated parameters, makes it attractive and quite useful to obtain a preliminary response. More important it can help in understanding physical phenomena, or identifying some interesting processes or results that are difficult to follow in numerical solutions. However, for realistic calculations it is necessary to use numerical methods where the media are usually extremely heterogeneous.

Appendix A. Symmetry Reduction Method of Advection-Diffusion Equation

Symmetry analysis has played an important role in obtaining exact solutions to PDEs. It has a long and extensive history and we refer the interested reader to some books (Bluman and Cole, 1974; Oliver, 1986; Hill, 1992) for detailed accounts.

In essence, the method for finding symmetry reductions of a given PDE is to find Lie group of infinitesimal transformations

$$\begin{bmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{t} \\ \tilde{w} \end{bmatrix} \to \begin{bmatrix} x \\ y \\ t \\ w \end{bmatrix} + \varepsilon \begin{bmatrix} \xi(x, y, t, u) \\ \zeta(x, y, t, u) \\ \tau(x, y, t, u) \\ \eta(x, y, t, u) \end{bmatrix}, \tag{A1}$$

which leaves the governing PDE (9) invariant. The first requirement of invariance implies that $\tilde{w}$ satisfies, as a function of $(\tilde{x}, \tilde{y}, \tilde{t})$, the same PDE (9) as w. We apply the algorithm that provides the symmetry algebra by constructing the second prolongation of the vector fields, that is, the differential operator of the form

$$\chi^2 = \xi\partial_x + \zeta\partial_y + \tau\partial_t + \eta\partial_w + \eta^t\partial_{w_t} + \eta^x\partial_{w_x} + \eta^{xx}\partial_{w_{xx}} + \eta^{yy}\partial_{w_{yy}}, \tag{A2}$$

where the functions η^t, η^x, η^{xx}, and η^{yy} are expressed in terms of ξ, ζ, τ, and η and their derivatives. The prolongation is then applied to Equation (9), and the resulting expression is required to vanish on solution of Equation (9). This leads to a set of determining equations that must then be solved. Solving this system of PDEs for ξ, ζ, τ, and η one obtains

$$\begin{aligned}
\xi &= a_1 + a_6 t - a_7 y + a_8 \frac{x}{2} + a_9 t x, \\
\zeta &= a_2 + a_5 t + a_7 x + a_8 \frac{y}{2} + a_9 t y, \\
\tau &= a_3 + a_8 t + a_9 t^2,
\end{aligned}$$

$$\eta = \frac{u}{2m}\left[a_4 - a_5 y + a_6(-x + nt) + a_7 ny + a_8\left(\frac{n}{2}x - \frac{n^2}{2}t\right) + \right.$$
$$\left. +a_9\left(-\frac{x^2+y^2}{2} + nxt - 2mt - \frac{n^2}{2}\right)\right]. \tag{A3}$$

The second requirement of invariance implies that w and $\tilde{w}$ are the same functions of their arguments. This gives a first order PDE as

$$\xi w_x + \zeta w_y + \tau w_t = \eta. \tag{A4}$$

This equation is solved by the use of the method of characteristics, which are given as solutions of any two ODEs obtained from

$$\frac{\mathrm{d}x}{\xi(x,y,t,w)} = \frac{\mathrm{d}y}{\zeta(x,y,t,w)} = \frac{\mathrm{d}t}{\tau(x,y,t.w)} = \frac{\mathrm{d}w}{\eta(x,y,t,w)}. \tag{A5}$$

The general solution of (A5) involves three constants, two of them; say $s(x, y, t)$ and $z(x, y, t)$ become new independent variables called similarity variables and the third constant; $F(s, z)$ plays the role of a new dependent variable, called similarity function. They reduce Equation (9) to PDE of two new variables s and z. To get the reduced ODEs, we apply once more the procedure mentioned above. It should be noted that similarity variables s, z and similarity function $F(s, z)$ obtained from (A5) are quite different from each other depending on the choice of the constants values $\{a_1, a_2, \ldots, a_9\}$ in (A3). We are able to distinguish five different types of reductions; from which the analytical solutions of Equation (9) may be constructed. It should be emphasized that, the above analysis of Equation (9) leading to (A3) is still valid for Equation (4). When $n = 0$ the exact solutions of Equation (4) are special cases of solutions for Equation (9).

A.1. THE FIRST TYPE SOLUTION

The PDE (9) of three independent variables can be reduced to PDE of two variables

$$F_s = mF_{zz} - \left(\frac{1}{2s} + \frac{n^2}{2m}\right)F, \tag{A6}$$

where $s = t, z = y$ and $w = \exp\left(-\frac{x^2}{4mt} + \frac{nx}{2m}\right)F(s, z)$ and subscripts in (A6) denote partial derivatives.

The PDE (A6) can be reduced to ODE:

$$\frac{\mathrm{d}G}{\mathrm{d}r} = -\left(\frac{1}{r} + \frac{n^2}{4m}\right)G, \tag{A7}$$

where $F(s, z) = e^{\frac{-z^2}{4mr}}G(r), r = s$. Equation (A7) has the solution

$$G(r) = \frac{A}{r}\exp\left(-\frac{n^2}{4m}\right). \tag{A8}$$

This solution leads by back substitutions to the following solutions of Equation (9):

$$w(x, y, t) = \frac{A}{t}\mathrm{Exp}\left(-\frac{x^2+y^2}{4mt} - \frac{n^2 t}{4m} + \frac{nx}{2m}\right), \tag{A9}$$

and for the special case, where $n = 0$, we have the following exact solution of Equation (4), as:

$$u(x, y, t) = \frac{A}{t}\mathrm{Exp}\left(-\frac{x^2+y^2}{4Mt}\right), \tag{A10}$$

where M and u are defined in Equation (4).

A.2. THE SECOND TYPE SOLUTION

Equation (9) can be written as a PDE of two new variables

$$F_s = mF_{zz} - nF_z - \frac{1}{2s}F, \tag{A11}$$

where $w(x, y, t) = e^{-y^2/4mt}F(s, z)$, $s = t$ and $z = x$.

Once more, the PDE (A11) can be reduced to ODE

$$\frac{\mathrm{d}G}{\mathrm{d}r} + \left[n - m + \frac{1}{2r}\right]G = 0, \tag{A12}$$

where $F(s, z) = e^z G(r)$, $r = s$. Equation (A12) has the solution

$$G(r) = \frac{A}{\sqrt{r}}e^{(m-n)r}. \tag{A13}$$

Back substitution leads to the analytic solution of Equation (9),

$$w(x, y, t) = \frac{A}{\sqrt{t}}\mathrm{Exp}\left(-\frac{y^2}{4mt} + (m-n)t + x\right). \tag{A14}$$

For the special case, $n = 0$, we have the exact solution of Equation (4), as

$$u(x, y, t) = \frac{A}{\sqrt{t}}\mathrm{Exp}\left(-\frac{y^2}{4Mt} + Mt + x\right). \tag{A15}$$

A.3. THE THIRD TYPE SOLUTION

Number of the independent variables in PDE (9) can be reduced, as

$$2m(F_{ss} + F_{zz}) + sF_s + zF_z + F = 0, \tag{A16}$$

where $w(x, y, t) = 1/\sqrt{t}F(s, z)$, $s = x/\sqrt{t}$ and $z = n\sqrt{t} - y/\sqrt{t}$.

Once more, by using symmetry reduction methods, we can reduce PDE (A16) to the confluent hypergeometric equation

$$r\frac{d^2G}{dr^2} + \left[1 + \frac{r}{4m}\right]\frac{dG}{dr} + \frac{G}{8m} = 0, \tag{A17}$$

where $F(s, z) = G(r)$ and $r = s^2 + z^2$. The Equation (A17) has the solution [21]

$$G(r) = A\Phi\left(\frac{1}{2}, 1; -\frac{r}{4m}\right) + B\Psi\left(\frac{1}{2}, 1; -\frac{r}{4m}\right), \tag{A18}$$

where Φ and Ψ are Kummer's functions, A and B are constants. Solutions (A18) lead by back substitution to the exact solution of Equation (9) in the form

$$w(x, y, t) = \frac{A}{\sqrt{t}}\Phi\left(\frac{1}{2}, 1; -\frac{x^2 + y^2}{4mt} - \frac{n^2 t}{4m} + \frac{nx}{2m}\right) \tag{A19}$$

and the exact solution of Equation (4) can be obtained by setting $n = 0$ in Equation (A19)

$$u(x, y, t) = \frac{A}{\sqrt{t}}\Phi\left(\frac{1}{2}, 1; -\frac{x^2 + y^2}{4Mt}\right). \tag{A20}$$

A.4. THE FOURTH TYPE SOLUTION

By the substitution $s = (x^2 + y^2)/2, z = t$ and $w(x, y, t) = e^{nx/2m}F(s, z)$ into Equation (9), we reduce the number of independent variables, to get

$$F_z = 2msF_{ss} + 2mF_s - \frac{n^2}{4m}F, \tag{A21}$$

Equation (A21) can be reduced to the modified Bessel's equation,

$$r\frac{d^2G}{dr^2} + \frac{dG}{dr} - \left(\frac{k}{2m} + \frac{n^2}{8m^2}\right)G = 0, \tag{A22}$$

where $F(s, z) = e^{-kz}G(r)$, $r = s$, and k is arbitrary constant and it has the general solution [21]

$$G(r) = AI_0(h\sqrt{r}) + BK_0(h\sqrt{r}), \tag{A23}$$

where $h^2 = (n^2 - 4km)/2m^2$, A and B are constants.

From (A23) and by back substitution, we get the following exact analytic solutions of Equation (9), as:

$$w(x, y, t) = e^{\frac{nx}{2m} - kt}\left[AI_0\left(h\sqrt{\frac{x^2 + y^2}{2}}\right) + BK_0\left(h\sqrt{\frac{x^2 + y^2}{2}}\right)\right]. \tag{A24}$$

The exact solution of Equation (4), can be obtained from Equation (A24) by setting $n = 0$.

$$u(x, y, t) = e^{-kt}\left[AI_0\left(h\sqrt{\frac{x^2 + y^2}{2}}\right) + BK_0\left(h\sqrt{\frac{x^2 + y^2}{2}}\right)\right], \tag{A25}$$

where $h^2 = \frac{2k}{M}$.

A.5. THE FIFTH TYPE OF SOLUTION

Following the same way, we get for Equation (9) the reduced form

$$m(F_{ss} + F_{zz}) - nF_z + kF_s = 0, \tag{A26}$$

where $s = y - kt$, $z = x$, $w(x, y, t) = F(s, z)$ and k is arbitrary constant.

Equation (A26) can be reduced, once more, to ODE

$$\frac{\mathrm{d}^2G}{\mathrm{d}r^2} + A\frac{\mathrm{d}G}{\mathrm{d}r} + BG = 0, \tag{A27}$$

where $G(r) = e^{-Cs}F(s, z)$, $r = z - s$ and the coefficients $C = (n - k)/2m$, $A = (k + n + 2mC)/2m$, $B = (mC^2 + kC)/2m$; and k is an arbitrary constant. The ODE (A27) has the solution

$$G(r) = c_1e^{ir} + c_2e^{jr}, \tag{A28}$$

where $i, j = 1/2(-A \pm \sqrt{A^2 - 4B})$; c_1 and c_2 are arbitrary constants. The formal solution (A28) leads by back substitution to the general solution of Equation (9) in the form

$$\begin{aligned} W(x, y, t) &= \exp\left(\frac{-Amx - ky + k^2t}{2m}\right) \times \\ &\quad \times [R \sinh p(x - y + kt) + Q \cosh p(x - y + kt)], \end{aligned} \tag{A29}$$

where $p = 1/2\sqrt{A^2 - 4B}$; R and Q are arbitrary constants. For $n = 0$, we get the solution of Equation (4) of the form

$$\begin{aligned} u(x, y, t) &= \exp\left(\frac{-ky + k^2t}{2M}\right) \times \\ &\quad \times [R \sinh p(x - y + kt) + Q \cosh p(x - y + kt)], \end{aligned} \tag{A30}$$

where $p = k/2\sqrt{2}M$.

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